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KINETIC THEORY OF THE CLASSICAL ELECTRON
GAS IN A POSITIVE BACKGROUND
I: EQUILIBRIUM THEORY

by

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ABSTRACT

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An approximation scheme is developed for calculating the equilibrium properties of a system containing both short range and weak long range interactions. The results are applied to the calculation of the pair correlation and equation of state for an electron gas in a neutralizing background. Although the starting point is the B-BG-K-Y hierarchy, the short-range divergence usually encountered in the "dynamic" approach is avoided, and the results (calculated through second order in the plasma parameter) are found to be in agreement with previous calculations by diagrammatic methods.

AUTHOR

1. Introduction

Most previous derivations^{1,2,3} of the equilibrium properties of plasma have relied on the diagrammatic techniques introduced by Mayer^{4,5}. While this approach has been elegantly systematized by Meeron¹, the physical motivation for the choice of diagrams seems unclear; furthermore it does not appear possible to generalize the method to non-equilibrium situations.

A second type of approach involves the approximate solution of coupled integro-differential equations (this is sometimes called the "equation of motion" method), and has been used quite successfully by Bogoliubov⁶ and others^{7,8,9}. However all attempts to carry the theory beyond first order in the plasma parameter have encountered a short-range divergence. Such divergences are of two types: (1) For a multicomponent plasma, the presence of attractive forces leads to a divergence unless bound states are taken into account. This difficulty seems never to have been adequately treated by any method (however cf. Theimer¹⁰). (2) Even for purely repulsive forces (e.g. the electron gas in a uniform positive background) a divergence occurs if one makes the usual assumption that the Coulomb potential is much less than the thermal energy. This is clearly a less fundamental difficulty, and is a consequence of the approximation method.

The present work employs a method similar to that of Bogoliubov in which divergences of type (2) do not occur. They are avoided by splitting the Coulomb potential up into two parts, one of which is weak (i.e. always $\ll kT$) and the other short range (compared to the inter-particle spacing).

We will show that such a division is always possible for systems with small plasma parameter.

The results include what is believed to be the first "dynamic" derivation of the equation of state for an electron gas which is correct through second order in the plasma parameter. In part II, a similar method will be used to derive a convergent kinetic equation for the electron gas.

2. General Formalism

While we are primarily interested in the electron gas in a neutralizing background, we avoid immediate specialization to this case so that the theory may also be applied to other systems (e.g. multi-component plasmas with repulsive cores). We suppose an arbitrary number of types with N_σ of type σ . The particles are numbered from 1 to $N \equiv \sum N_\sigma$, and the i^{th} particle is specified by its position \underline{R}_i and its type σ_i . We shall use the shorthand $\chi_i \equiv (\underline{R}_i, \sigma_i)$.

The class of systems under consideration are those for which the interaction between any two particles of the same or different types may be split into two parts ψ and ϕ such that

$$\psi(|\underline{R}|, \sigma, \tau) = 0, \quad |\underline{R}| > a_{\sigma\tau};$$

$$\frac{4\pi a_{\sigma\tau}^3}{v} \ll 1, \quad \text{all } \sigma, \tau; \quad (1)$$

where

$$v = \lim_{\substack{N \rightarrow \infty \\ V \rightarrow \infty}} \frac{V}{N} \quad (2)$$

and

$$\frac{\phi(|\underline{R}|, \sigma, \tau)}{\theta} < 1, \quad \text{all } |\underline{R}|, \sigma, \tau, \quad (3)$$

where θ is the temperature in energy units.

The spatial s-particle distribution function is defined by

$$n_s(x_1 \dots x_s) = \frac{V^s \prod_{i=1}^s N_{\sigma_i} \int \dots \int d\underline{R}_{s+1} \dots d\underline{R}_N \exp\left\{-\frac{\Psi_N - \Phi_N}{\theta}\right\}}{N^s \int \dots \int d\underline{R}_1 \dots d\underline{R}_N \exp\left\{-\frac{\Psi_N - \Phi_N}{\theta}\right\}} \quad (4)$$

where

$$\Psi_N = \sum_{i < j \leq N} \sum \psi(|\underline{R}_i - \underline{R}_j|, \sigma_i, \sigma_j) \quad (5)$$

$$\Phi_N = \sum_{i < j \leq N} \sum \phi(|\underline{R}_i - \underline{R}_j|, \sigma_i, \sigma_j) \quad (6)$$

Taking the gradient of (4) with respect to the position of the i^{th} particle and letting $V \rightarrow \infty$, $N \rightarrow \infty$, V/N finite, one finds the hierarchy of equations

$$\nabla_i n_s + \frac{n_s}{\theta} \nabla_i (\Phi_s + \Psi_s) + \frac{1}{\theta V} \int d\underline{x} n_{s+1}(\underline{x}) \nabla_i \left[\psi(\underline{x}_i, \underline{x}) + \phi(\underline{x}_i, \underline{x}) \right] = 0 \quad (7)$$

Here we have suppressed the arguments $x_1 \dots x_s$, and used the shorthand

$$\int d\underline{x} \equiv \int d\underline{R} \sum_{\sigma} \quad (8)$$

It is useful to divide out a Boltzmann factor for the ψ - potential,
i.e. we set

$$n_s = m_s e^{-\psi_s/\theta} \quad (9)$$

to obtain

$$\begin{aligned} \nabla_1 m_s + \frac{m_s}{\theta} \nabla_1 \phi_s + \frac{1}{\theta v} \int d\chi m_{s+1}(\chi) \left(1 + f_s(\chi) \right) \nabla_1 \phi(\chi_1, \chi) \\ - \frac{1}{v} \int d\chi m_{s+1}(\chi) \nabla_1 f_s(\chi) = 0 \end{aligned} \quad (10)$$

where

$$\begin{aligned} f_s(\chi) &\equiv f_s(\chi_1 \dots \chi_s; \chi) = e^{-\psi_s/\theta} - 1 \\ &= \prod_{i=1}^s \left[1 + f(\chi_i, \chi) \right] - 1 \end{aligned} \quad (11)$$

and

$$f(\chi_1, \chi) \equiv f_1(\chi_1; \chi) = e^{-\psi(\chi_1, \chi)/\theta} - 1 \quad (12)$$

is the function introduced by Mayer. We note that

$$f(\chi_1, \chi) = 0 \quad |R_1 - R| > a_{\sigma_1 \sigma} \quad (13)$$

As boundary conditions we must require that the correlation between the particle positions must vanish as the particle separations go to ∞ . In our normalization this is equivalent to

$$\lim_{\text{all } |R_1 - R_j| \rightarrow \infty} m_s = \prod_{i=1}^s \left(\frac{N_{\sigma_i}}{N} \right) \quad (14)$$

The next three sections are devoted to an approximation scheme for solving (10) with the boundary conditions (14) and the basic assumptions (1), (3).

3. Ordinary Gas Mixtures

In order to illustrate the methods we will first consider two special cases. The first of these is the case of ordinary gas mixtures, for which we may take ϕ to be identically zero (i.e. the entire potential is short range). Then (10) is simplified to

$$\nabla_i m_s = \frac{1}{v} \int dX m_{s+1}(X) \nabla_i f_s(X) \quad (15)$$

But $f_s(X)$ vanishes if

$$|\underline{R} - R_i| > a_{\sigma\sigma i} \quad 1 \leq i \leq s$$

It follows that the right hand side will be smaller than the left by a factor of order $\frac{4\pi a^3}{v}$ where a is the largest of the $a_{\sigma\tau}$ (according to (1) this factor will be small). This suggests an expansion of the

m_s in powers of the density corresponding to the usual virial expansion i.e.

$$m_s = \sum_{n=0}^{\infty} \left(\frac{1}{v} \right)^n m_s^{(n)} \quad (16)$$

Substituting (16) in (15) and equating like powers of $\left(\frac{1}{v} \right)$, one finds

$$\nabla_i m_s^{(n)} = \int dX m_{s+1}^{(n-1)}(X) \nabla_i f_s(X) \quad (17)$$

In c^{th} order one has

$$\nabla_i m_s^{(0)} = 0$$

or

$$m_s^{(0)} = \text{const} = \prod_{i=1}^s \left(\frac{N_{\sigma_i}}{N} \right) \quad (18)$$

where the constant has been determined from (14).

The first order equation is

$$\nabla_i m_s^{(1)} = m_s^{(0)} \int d\chi \left(\frac{N_{\sigma}}{N} \right) \nabla_i f_s(\chi) \quad (19)$$

or

$$m_s^{(1)} = m_s^{(0)} \int d\chi \left(\frac{N_{\sigma}}{N} \right) f_s(\chi) + \text{const} \quad (20)$$

The constant is again determined by the boundary condition, which requires

$$\lim_{\text{all } R_{ij} \rightarrow \infty} m_s^{(1)} = 0 \quad (21)$$

The limit of the integral in (20) is determined by using the expansion

(11) for f_s in terms of the Mayer f 's and observing that terms containing two or more Mayer functions will give no contribution to the integral in the limit as all $R_{ij} \rightarrow \infty$. Thus

$$\lim_{\text{all } R_{ij} \rightarrow \infty} \int d\chi \frac{N_{\sigma}}{N} f_s(\chi) = \sum_{i=1}^s \int d\chi \frac{N_{\sigma}}{N} f(\chi_i, \chi) \quad (22)$$

and

$$m_s^{(1)} = m_s^{(0)} \int d\chi \frac{N_{\sigma}}{N} \left[f_s(\chi_1 \dots \chi_s; \chi) - \sum_{i=1}^s f(\chi_i, \chi) \right] \quad (23)$$

The second order calculation, of which we omit the details, gives

$$\begin{aligned}
 m_s^{(2)} = & \frac{m_s^{(0)}}{2} \left\{ \left[\int d\mathbf{x} \frac{N}{N} \left(f_s(\mathbf{x}) - \sum_{i=1}^s f(\mathbf{x}_i, \mathbf{x}) \right) \right]^2 \right. \\
 & + \int d\mathbf{x} \int d\mathbf{x}' \left(\frac{N}{N} \right) \left(\frac{N}{N} \right) f(\mathbf{x}, \mathbf{x}') \left[f_s(\mathbf{x}) f_s(\mathbf{x}') \right. \\
 & \left. \left. - \sum_{i=1}^s f(\mathbf{x}_i, \mathbf{x}) f(\mathbf{x}_i, \mathbf{x}') \right] \right\} \quad (24)
 \end{aligned}$$

One could go on in this fashion to any desired order in the small parameter, although it seems difficult to calculate the general term (however cf. Cohen¹¹).

4. Weak Interactions

The second special case we will consider is the class of systems for which the interaction potential is $\ll \theta$ for all particle separations. While such systems appear to be rare, their consideration will prove useful in understanding systems, such as a plasma, where the interaction is weak through most of its range.

In this weak interaction limit we may set the ψ 's and f 's equal to zero to obtain

$$\nabla_i m_s + \frac{m_s}{\theta} \nabla_i \phi_s + \frac{1}{\theta v} \int d\mathbf{x} m_{s+1}(\mathbf{x}) \nabla_i \phi(\mathbf{x}_i, \mathbf{x}) = 0 \quad (25)$$

In view of our assumption (3) about the nature of the potential, the second term will be smaller than the first by at least a factor of ϕ_{\max}/θ . On the other hand the integral term may be estimated as the gradient of a function which has the order of magnitude

$$\frac{4 \pi \langle \phi \rangle R_0^3}{\theta v}$$

where $\langle \phi \rangle$ is the average potential and R_0 is the range (as a function

of $\frac{1}{R}$ of $\phi(x_1, x)$ or $m_{s+1}(x_1 \dots x_s, x)$, whichever is smaller.

For a plasma, R_0 is the Debye length, and

$$\langle \phi \rangle = \frac{e^2}{R_0}$$

thus the integral term in (25) is the same order of magnitude as the first term. Thus the situation for weak, long range forces is exactly the reverse of that for the short range interactions discussed in Section 3.

These considerations suggest that a suitable approximation scheme for the solution of (25) would be to formally introduce a parameter ϵ in front of the second term, expand m_s in powers of ϵ , and equate like powers. The only difficulty with this approach is that the resulting equations relate a given approximation for m_s to m_{s+1} in the same approximation. Therefore, in order to insure a determinate set of equations, it is convenient to introduce a second parameter (say δ) as the coefficient of the integral term. Since, as noted above, the integral term will not be small in general, we may not treat δ as a small parameter, and must include all powers of δ for each power of ϵ . Physically, this corresponds to considering, instead of a Coulomb potential, a Yukawa potential ($\frac{e^{-\alpha R}}{R}$ dependence) with α large enough to ensure convergence of the infinite sums which occur; the results are then analytically continued down to $\alpha = 0$ (this notion was first introduced by Mayer⁵). In this sense, as a limit of the results for a screened potential, the results will be unique.

Accordingly we rewrite (25) as

$$\nabla_i m_s + \frac{\epsilon m_s}{\theta} \nabla_i \phi_s + \frac{\delta}{\theta v} \int dx m_{s+1}(x) \nabla_i \phi(x_1, x) = 0 \quad (25')$$

expand m_s

$$m_s = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \epsilon^p \delta^q m_s^{(p,q)} = \sum_{p=0}^{\infty} \epsilon^p m_s^{(p)} \quad (26)$$

and equate like powers to obtain

$$\nabla_i m_s^{(p,q)} + \frac{m_s^{(p-1,q)}}{\theta} \nabla_i \phi_s + \frac{1}{\theta v} \int dX m_{s+1}^{(p,q-1)}(X) \nabla_i \phi(X_1, X) = 0 \quad (27)$$

For bounday conditions, we require

$$\lim_{\text{all } R_{ij} \rightarrow \infty} m_s^{(0,0)} = \prod_{i=1}^s \left(\frac{N_{\sigma_i}}{N} \right); \quad \lim_{\text{all } R_{ij} \rightarrow \infty} m_s^{(p,q)} = 0, \quad p \text{ or } q > 0 \quad (28)$$

For $p=0$, one has

$$\nabla_i m_s^{(0,0)} = 0 \quad (29)$$

$$\nabla_i m_s^{(0,q)} = - \frac{1}{\theta v} \int dX m_{s+1}^{(0,q-1)} \nabla_i \phi(X_1, X) \quad (30)$$

The solution of (29) with (28) is

$$m_s^{(0,0)} = \prod_{i=1}^s \frac{N_{\sigma_i}}{N}, \quad (31)$$

and (30) with $q=1$ becomes

$$\nabla_i m_s^{(0,1)} = - \frac{m_s^{(0,0)}}{\theta v} \int dX \frac{N_{\sigma}}{N} \nabla_i \phi(X_1, X) = 0 \quad (32)$$

where the last step follows because $\phi(X_1, X)$ depends on $\underline{R}_1, \underline{R}$ only

through $|\underline{R}_i - \underline{R}|$. In view of (28), (32) implies

$$m_s^{(0,1)} = 0 \quad (33)$$

and it immediately follows from (28) and (30) that

$$m_s^{(0,q)} = 0 \quad q > 0 \quad (34)$$

Turning to $p=1$, one finds

$$\nabla_i m_s^{(1,0)} + \frac{m_s^{(0,0)}}{\theta} \nabla_i \phi_s = 0 \quad (35)$$

and

$$\nabla_i m_s^{(1,q)} + \frac{1}{\theta v} \int d\mathbf{x} m_{s+1}^{(1,q-1)}(\mathbf{x}) \nabla_i \phi(\mathbf{x}_1, \mathbf{x}) = 0 \quad (36)$$

where we have used (34). Thus

$$m_s^{(1,0)} = - \frac{m_s^{(0,0)}}{\theta} \phi_s \quad (37)$$

(this would be the leading significant term if ϕ were both weak and short range). Substituting (37) into (36) with $q=1$, and noting that (cf. (6))

$$\phi_{s+1}(\mathbf{x}_1 \dots \mathbf{x}_s, \mathbf{x}) = \phi_s(\mathbf{x}_1 \dots \mathbf{x}_s) + \sum_{j=1}^s \phi(\mathbf{x}_j, \mathbf{x}), \quad (38)$$

one finds

$$\nabla_i m_s^{(1,1)} = \frac{m_s^{(0,0)}}{\theta v} \int d\mathbf{x} \frac{N_\sigma}{N} \sum_{j=1}^s \phi(\mathbf{x}_j, \mathbf{x}) \nabla_i \phi(\mathbf{x}_1, \mathbf{x}) \quad (39)$$

which has the solution

$$m_s^{(1,1)} = \frac{m_s^{(0,0)}}{\theta^2 v} \int d\mathbf{x} \frac{N_\sigma}{N} \sum_{i < j \leq s} \phi(\mathbf{x}_i, \mathbf{x}) \phi(\mathbf{x}, \mathbf{x}_j) = - \frac{m_s^{(0,0)}}{\theta} \phi_s^{(1)} \quad (40)$$

where we have introduced the notation

$$\Phi_s^{(n)} = \left(-\frac{1}{\theta v} \right)^n \sum_{i < j \leq s} \sum \Phi_n(x_i, x_j) ; \quad (41)$$

$$\Phi_n(x_i, x_j) = \int dx^{(1)} \dots dx^{(n)} \prod_{k=1}^n \left(\frac{N_{\sigma(k)}}{N} \right)$$

$$\Phi(x_i, x^{(1)}) \Phi(x^{(1)}, x^{(2)}) \dots \Phi(x^{(n)}, x_j), \Phi_0 = \Phi. \quad (42)$$

It is easy to show that

$$m_s^{(1,2)} = - \frac{m_s^{(0,0)}}{\theta} \Phi_s^{(2)} \quad (43)$$

and in general (as may be proved by induction or verified by direct substitution into (36))

$$m_s^{(1,q)} = - \frac{m_s^{(0,0)}}{\theta} \Phi_s^{(q)} \quad (44)$$

The total contribution to first order in the ratio of potential to thermal energy is given by summing over q :

$$m_s^{(1)} = \sum_{q=0}^{\infty} m_s^{(1,q)} = - m_s^{(0,0)} \left(\frac{D}{\theta} \frac{\Phi_s}{\theta} \right) \quad (45)$$

The quantity

$$\Phi_s^{(D)} \equiv \sum_{q=0}^{\infty} \Phi_s^{(q)} \equiv \sum_{i < j \leq s} \Phi_D(x_i, x_j) \quad (46)$$

is the summed contribution of all "chain type" interactions in which each particle is interacting with only two others. This effectively replaces the (possibly) infinite range potential Φ_s by a shielded potential.

If Φ is the Coulomb potential, it is easy to show that Φ_D will be the Debye potential. In Section 6 we will calculate Φ_D for a slightly different choice of Φ . It should be remarked that (45) could also have been obtained by omitting the δ expansion and introducing the Ansatz

$$m_s^{(1)} = \sum_{i < j \leq s} m_2^{(1)}(x_i, x_j) \prod_{k \neq i, j}^s m_1^{(0)}(\sigma_k) \quad (47)$$

In our treatment, (47) appears as a consequence rather than an additional assumption.

For $p=2$, (27) becomes

$$\nabla_i m_s^{(2,q)} = m_s^{(0,0)} \left(\frac{\Phi_s^{(q)}}{\theta} \right) \nabla_i \Phi_s + \frac{1}{\theta v} \int dx m_{s+1}^{(2,q-1)}(x) \nabla_i \Phi(x_i, x) = 0 \quad (48)$$

The algebra involved in solving these equations is considerably more involved than that of the first order equations, so we will here merely state the result (cf. work in footnote 7 for details):

$$\begin{aligned}
m_s = \sum_{q=0}^{\infty} m_s^{(2,q)} = m_s^{(0,0)} & \left\{ \frac{1}{2} \left(\frac{\phi_s^D}{\theta} \right)^2 \right. \\
- \frac{1}{v\theta^3} \int dx \frac{N_\sigma}{N} & \left[\sum_{i < j < k \leq s} \phi_D(x_i, x) \phi_D(x_j, x) \phi_D(x_k, x) \right. \\
+ \frac{1}{2} \sum_{i \neq j} \phi_D^2(x_i, x) & \left. \phi_D(x_j, x) \right] \\
+ \frac{1}{2\theta^4} \frac{1}{v^2} \int dx \int dx & \left[\phi_D(x, x) \right]^2 \sum_{i < j \leq s} \phi_D(x_i, x) \phi_D(x, x_j) \left. \right\} \quad (49)
\end{aligned}$$

as expected, (49) and (45) depend on ϕ only through the shielded potential ϕ_D . As in the short range case, we will not go beyond second order.

5. The General Case

We now return to the general situation where neither ψ nor ϕ are zero, rewrite (10) as

$$\begin{aligned}
\nabla_1 m_s &= \frac{1}{v} \int dx m_{s+1}(x) \nabla_1 f_s(x) + \frac{m_s}{\theta} \nabla_1 \phi_s \\
+ \frac{1}{\theta v} \int dx f_s(x) m_{s+1}(x) & \nabla_1 \phi(x_1, x) \\
+ \frac{1}{\theta v} \int dx m_{s+1}(x) & \nabla_1 \phi(x_1, x) = 0 \quad (10)
\end{aligned}$$

and study it term by term. Just as in Section 3 and 4 we estimate the relative magnitudes of the first three terms as

$$1 : 4\pi a^3/v : \phi_{\max}/\theta$$

Because of the factor $f_s(x)$, the integrand of the fourth term will have a range of order a , and its relative magnitude may be estimated as $(4\pi a^3/v) \cdot (\langle \phi \rangle / \theta)$, i.e. proportional to both small parameters. The last term is more difficult, since it is sensitive to the range of $m_{s+1}(x)$. There will be some terms in $m_{s+1}(x)$ which contain ψ in such a way that their range is of order a ; their contribution to the fifth term will again be proportional to both small parameters. On the other hand, there will be some terms (such as those independent of ψ) which may have a range $\gtrsim \left(\frac{v \theta}{\langle \phi \rangle} \right)^{1/3}$; their contribution will be $O(1)$. Thus it is convenient to split the function $m_{s+1}(x_1 \dots x_s, x)$ into two parts:

$$m_{s+1}(x) = \bar{m}_{s+1}(x) + \tilde{m}_{s+1}(x) \quad (50)$$

where $\bar{m}_{s+1}(x)$ consists of all terms in $m_{s+1}(x)$ which vanish for

$$|R_i - \bar{R}| > b, \quad i = 1 \dots s; \quad \frac{4\pi b^3}{v} \ll 1,$$

and $\tilde{m}_{s+1}(x)$ contains all the remaining terms. (This division may be made unique by expressing the results in terms of the usual graphs; the essential point is that \bar{m}_{s+1} contains all contributions whose range is determined by the range of ψ).

We will now introduce three formal parameters into (10):

λ , the "range parameter" for ψ , and ϵ and δ , which have the same significance as in Section 4 (it is not convenient to use $(1/v)$)

as a parameter as in Section 3, since it occurs in at least one place where it may not be treated as small). In view of the above discussion of the relative magnitude of the terms in (10), we may write it as

$$\begin{aligned} \nabla_i m_s &= \frac{\lambda}{v} \int d\chi \, m_{s+1}(\chi) \nabla_i f_s(\chi) + \frac{\epsilon m_s}{\theta} \nabla_i \phi_s \\ &+ \frac{\lambda \epsilon}{\theta v} \int d\chi \left[f_s(\chi) m_{s+1}(\chi) + \bar{m}_{s+1}(\chi) \right] \nabla_i \phi(\chi_1, \chi) \\ &+ \frac{\delta}{\theta v} \int d\chi \, \bar{m}_{s+1}(\chi) \nabla_i \phi(\chi_1, \chi) = 0 ; \end{aligned} \quad (10')$$

we reiterate that the introduction of the formal parameters in (10') is a way of noting that the relative magnitudes of the terms are

$$1: \left(\frac{4\pi a^3}{v} \right) : \frac{\phi_{\max}}{\theta} : \left(\frac{4\pi a^3}{v} \right) \left(\frac{\langle \phi \rangle}{\theta} \right) : \frac{\langle \phi \rangle}{\theta} \left(\frac{4\pi R_0^3}{v} \right)$$

where a is the range of ψ , and R_0 is the range of ϕ_D . We then make a triple expansion of m_s

$$\begin{aligned} m_s &= \sum \sum \sum m_s^{(n,p,q)} \lambda^n \epsilon^p \delta^q \\ &= \sum \sum m_s^{(n,p)} \lambda^n \epsilon^p \end{aligned} \quad (51)$$

and equate like powers to find

$$\begin{aligned} \nabla_i m_s^{(n,p,q)} &= \frac{1}{v} \int d\chi \, m_{s+1}^{(n-1,p,q)}(\chi) \nabla_i f_s(\chi) \\ &+ \frac{m_s^{(n,p-1,q)}}{\theta} \nabla_i \phi_s + \frac{1}{\theta v} \int d\chi \left[f_s(\chi) m_{s+1}^{(n-1,p-1,q)}(\chi) \right. \\ &+ \left. \bar{m}_{s+1}^{(n-1,p-1,q)}(\chi) \right] \nabla_i \phi(\chi_1, \chi) \\ &+ \frac{1}{\theta v} \int d\chi \, \bar{m}_{s+1}^{(n,p,q-1)}(\chi) \nabla_i \phi(\chi_1, \chi) = 0 \end{aligned} \quad (52)$$

For $p=q=0$, the equations are the same as those in Section 3, save for the factor of $\frac{1}{v}$ in front of the integral. Thus we may immediately write

$$m_s^{(0,0,0)} = \prod_{i=1}^s \frac{N_{\sigma_i}}{N} \quad (53)$$

$$m_s^{(1,0,0)} = \frac{m_s^{(0,0,0)}}{v} \int dx \frac{N_{\sigma}}{N} \left[f_s(x) - \sum_{i=1}^s f(x_i, x) \right] \quad (54)$$

$$\begin{aligned} m_s^{(2,0,0)} &= \frac{m_s^{(0,0,0)}}{2v^2} \left\{ \left[\int dx \frac{N_{\sigma}}{N} \left(f_s(x) - \sum_{i=1}^s f(x_i, x) \right) \right]^2 \right. \\ &+ \int dx \int dx' \left(\frac{N_{\sigma}}{N} \right) \left(\frac{N_{\sigma}}{N} \right) f(x, x') \left[f_s(x) f_s(x') \right. \\ &\left. \left. - \sum_{i=1}^s f(x_i, x) f(x_i, x') \right] \right\} \quad (55) \end{aligned}$$

Just as in Section 4, one easily shows that

$$m_s^{(0,0,q)} = 0 \quad q > 0. \quad (56)$$

Further, since $m_{s+1}^{(1,0,0)}(x)$ and $m_{s+1}^{(2,0,0)}(x)$ contain only short range contributions which vanish for $|\underline{R}_1 - \underline{R}| > 2a$ (they depend only on ψ , not ϕ),

$$\widetilde{m}_{s+1}^{(1,0,0)}(x) = \widetilde{m}_{s+1}^{(2,0,0)}(x) = 0, \quad (57)$$

and

$$m_s^{(1,0,q)} = m_s^{(2,0,q)} = 0 \quad q > 0 \quad (58)$$

On the other hand, if $n=0$, the equations are independent of ψ , thus

$$\tilde{m}_{s+1}^{(0,p,q)}(x) = m_{s+1}^{(0,p,q)}(x) \quad (59)$$

and the equations and results reduce to those of Section 4 [equations (44), (41), (42), (49)] .

If we assume that the two small parameters are the same order of magnitude, and restrict ourselves to contributions up to second order, the only new equations are for $n=p=1$:

$$\begin{aligned} \nabla_1 m_s^{(1,1,0)} + \frac{m_s^{(0)}}{v\theta} \int dx \frac{N_\sigma}{N} \left[\phi_s + \sum_{j=1}^s \phi(x_j, x) \right] \\ \nabla_1 f_s(x) + \frac{m_s^{(0)}}{v\theta} \int dx \frac{N_\sigma}{N} \left[f_s(x) - \sum_{j=1}^s f(x_j, x) \right] \nabla_1 \phi_s \\ + \frac{m_s^{(0)}}{v} \int dx \frac{N_\sigma}{N} f_s(x) \nabla_1 \phi(x_1, x) = 0 \end{aligned} \quad (60)$$

and

$$\begin{aligned} \nabla_1 m_s^{(1,1,q)} + \frac{m_s^{(0)}}{v\theta} \int dx \frac{N_\sigma}{N} \left[\phi_s^{(q)} + \left(-\frac{1}{v\theta} \right)^q \sum_{j=1}^s \phi_q(x_j, x) \right] \\ \nabla_1 f_s(x) + \frac{1}{v\theta} \int dx \tilde{m}_{s+1}^{(1,1,q-1)}(x) \nabla_1 \phi(x_1, x) = 0 \quad q > 0 \end{aligned} \quad (61)$$

where we have used (53), (54), (55), (58), (44), (41), (42), (38), and (49) for $m_s^{(1,0,q)}$, $m_s^{(0,1,q)}$ and $m_s^{(0,0,q)}$ plus the fact that

$$\tilde{m}_s^{(0,0,0)} = 0$$

($m_s^{(0,0,0)} = m_s^{(0)}$ has infinite range).

is evident that (60) is of the form

$$\nabla_i F(x_1 \dots x_s) = 0 ;$$

it is only necessary to determine the constant so that the boundary condition

$$\lim_{\text{all } R_{ij} \rightarrow \infty} m_s^{(1,1,0)} = 0$$

is satisfied. The result is

$$m_s^{(1,1,0)} = - \frac{m_s^{(0)}}{v \theta} \int d\chi \quad \frac{N_\sigma}{N} \left\{ \phi_s \left[f_s(\chi) - \sum_{i=1}^s f(x_i, \chi) \right] \right. \\ \left. + f_s(\chi) \sum_{i=1}^s \phi(x_i, \chi) - \sum_{i=1}^s f(x_i, \chi) \phi(x_i, \chi) \right\} \quad (62)$$

We may expect that $m_s^{(1,1,q)}$ will contain terms like those in $m_s^{(1,1,0)}$, with ϕ replaced by the convoluted potential $\left(-\frac{1}{\theta v}\right)^q \phi_q$. Thus it is useful to make the substitution

$$m_s^{(1,1,q)} = k_s^{(1,1,q)} + l_s^{(1,1,q)} \quad (63)$$

where

$$k_s^{(1,1,q)} = - \frac{m_s^{(0)}}{v \theta} \int d\chi \quad \frac{N_\sigma}{N} \left\{ \phi_s^{(q)} \left[f_s(\chi) - \sum_{i=1}^s f(x_i, \chi) \right] \right. \\ \left. + f_s(\chi) \left(-\frac{1}{v \theta}\right)^q \sum_{i=1}^s \phi_q(x_i, \chi) - \left(-\frac{1}{v \theta}\right)^q \sum_{i=1}^s \phi_q(x_i, \chi) f(x_i, \chi) \right\} \quad (64)$$

The criterion for calculating the long range part $\tilde{k}_{s+1}^{(1,1,0)}(\chi)$

is best expressed in terms of graphs. We associate a point with each fixed variable x_i or integration variable χ, χ' , etc.

a factor $\phi(x, x')$ is represented by a solid line between the points x, x' , and a factor $f(x, x')$ by a dotted line (f bond). It is then clear that the terms which contribute to k_{s+1} (or m_{s+1}) will be those for which each path between the point x and any of the points $x_1 \dots x_s$ contains at least one ϕ bond (this includes those terms independent of x , where the point x is not connected to any of the points x_i in the corresponding graph). It follows that

$$\begin{aligned} \tilde{k}_{s+1}(x) &= \frac{N_\sigma}{N} k_s \\ &+ \left(-\frac{1}{v\theta}\right)^{q+1} m_s^{(0)} \frac{N_\sigma}{N} \int dx' \frac{N_{\sigma'}}{N} \left\{ \left[f_s(x') - \sum_{i=1}^s f(x_i, x') \right] \right. \\ &\quad \sum_{j=1}^s \phi_q(x_j, x) + f(x, x') \sum_{i=1}^s \phi_q(x_i, x') \\ &\quad \left. + f_s(x') \phi_q(x, x') \right\} \end{aligned} \quad (65)$$

where we have used

$$\phi_{s+1}^{(q)}(x_1 \dots x_s, x) = \phi_s^{(q)}(x_1 \dots x_s) + \left(-\frac{1}{v\theta}\right)^q \sum_{i=1}^s \phi_q(x_i, x) \quad (66)$$

and

$$f_{s+1}(x_1 \dots x_s, x; x') = f_s(x') \left[1 + f(x, x') \right] + f(x, x') \quad (67)$$

Substituting (63) and (65) into (61), one finds

$$\begin{aligned} \nabla_i l_s^{(1,1,q)} - \left(-\frac{1}{v\theta}\right)^{q+1} m_s^{(0)} \int d\chi \frac{N_\sigma}{N} \int d\chi' \frac{N_{\sigma'}}{N} f(\chi, \chi') \\ \sum_{j=1}^s \phi_{q-1}(\chi_j, \chi') \nabla_i \phi(\chi_i, \chi) \\ + \frac{1}{v\theta} \int d\chi l_{s+1}^{(1,1,q-1)}(\chi) \nabla_i \phi(\chi_i, \chi) = 0 \end{aligned} \quad (68)$$

Since $m_s^{(1,1,0)} = k_s^{(1,1,0)}$,

$$l_s^{(1,1,0)} = 0; \quad (69)$$

it is easy to show that

$$l_s^{(1,1,q)} = \left(-\frac{1}{v\theta}\right)^{q+1} m_s^{(0)} \int d\chi \frac{N_\sigma}{N} \int d\chi' \frac{N_{\sigma'}}{N} f(\chi, \chi')$$

$$\sum_{i < j \leq s} \sum_{q_1 + q_2 = q-1} \phi_{q_1}(\chi_i, \chi) \phi_{q_2}(\chi, \chi_j) \quad q \geq 1 \quad (70)$$

Summing (63), (64), (70) over q gives the result to first order in each small parameter:

$$\begin{aligned} m_s^{(1,1)} = \sum_{q=0}^{\infty} m_s^{(1,1,q)} = -\frac{m_s^{(0)}}{v\theta} \int d\chi \frac{N_\sigma}{N} \left\{ \phi_s^D \left[f_s(\chi) \right. \right. \\ \left. \left. - \sum_{i=1}^s f(\chi_i, \chi) \right] + f_s(\chi) \sum_{i=1}^s \phi_D(\chi_i, \chi) \right\} \end{aligned}$$

$$\begin{aligned}
& - \sum_{i=1}^s f(x_i, x) \phi_D(x_i, x) - \frac{1}{v\theta} \int dx' \frac{N_{\sigma'}}{N} f(x, x') \\
& \left. \sum_{i < j \leq s} \phi_D(x_i, x) \phi_D(x', x_j) \right\} \quad (71)
\end{aligned}$$

This completes the calculations through second order. In the next two sections we will apply them to the calculation of the equation of state for an electron gas in a positive background.

6. The Electron Gas; The Screened Potential

As mentioned in the introduction, the main weakness of previous dynamic derivations of the equilibrium properties of the electron gas is the assumption that the potential is weak compared to the thermal energy, which clearly cannot be satisfied everywhere for point electrons. In this section we show how the Coulomb potential can be broken into two parts which satisfy respectively the short range condition (1) and the "weakness" restriction (3). One such division is (we now specialize to one type - electrons with charge $(-e)$)

$$\psi(R) = \frac{e^2}{R} \frac{S(a-R)}{R} \quad (72)$$

$$\phi(R) = \frac{e^2}{R} \frac{S(R-a)}{R} \quad (73)$$

where $S(z)$ is the unit step function. It is only necessary to show that we can find an a such that

$$\frac{4\pi a^3}{v} \ll 1 \quad (74)$$

$$\frac{e^2}{a\theta} \ll 1 \quad (75)$$

which insures that (1) and (3) will be satisfied. Equations (74) and (75) may be combined as

$$\frac{e^2}{\theta} \ll a \ll \left(\frac{v}{4\pi} \right)^{1/3} \quad (76)$$

Such a choice of \underline{a} will be possible provided

$$\frac{4\pi \left(\frac{e^2}{\theta} \right)^3}{v} \ll 1 \quad (77)$$

But this is equivalent to the usual condition that the number of particles in a Debye sphere should be large, or that the so called plasma parameter is small, and will always be assumed. The results will clearly not be sensitive to the choice of \underline{a} , nor indeed to the particular way in which the potential is divided; for definiteness we choose

$$a = \sqrt{\frac{e^2}{\theta \kappa}} \quad (78)$$

where

$$\kappa = \sqrt{\frac{4\pi e^2}{\theta v}} \quad (79)$$

is the reciprocal Debye length. For this choice, the two small parameters are equal:

$$\frac{4\pi a^3}{v} = \frac{e^2}{\theta a} = \sqrt{\frac{e^2 \kappa}{\theta}} = \sqrt{\epsilon} \quad (80)$$

where ϵ is the real small parameter of the problem, the plasma parameter.

We now proceed to specialize the results of the preceding sections to this case; ultimately we will find the equation of state through second order in ϵ . The first step is to calculate the shielded potential

ϕ_D corresponding to ϕ . This is given by

$$\phi_D(R) = \sum_{q=0}^{\infty} \left(-\frac{1}{v\theta} \right)^q \phi_q(R) \quad (81)$$

But

$$\begin{aligned} \phi_q(R) &= \int d\vec{R}_1 \dots d\vec{R}_q \phi(R_1) \phi(R_2) \dots \phi(R_q) \phi\left(\left|\vec{R} - \sum_{n=1}^q \vec{R}_n\right|\right) \\ &= \frac{1}{(2\pi)^3} \int d\vec{k} e^{-i\vec{k}\cdot\vec{R}} (\phi(\vec{k}))^{q+1} \end{aligned} \quad (82)$$

where

$$\phi(\vec{k}) = \int d\vec{R} e^{i\vec{k}\cdot\vec{R}} \phi(|\vec{R}|) \quad (83)$$

so

$$\begin{aligned} \phi_D(R) &= \frac{1}{(2\pi)^3} \int d\vec{k} \phi(\vec{k}) \sum_{q=0}^{\infty} \left(\frac{-\phi(\vec{k})}{v\theta} \right)^q e^{-i\vec{k}\cdot\vec{R}} \\ &= \frac{1}{(2\pi)^3} \int d\vec{k} \frac{\phi(\vec{k}) e^{-i\vec{k}\cdot\vec{R}}}{\left[1 + \frac{\phi(\vec{k})}{v\theta} \right]} \end{aligned} \quad (84)$$

where the convergence of the geometric series under the integral sign has been assumed (actually one should put a factor $e^{-\alpha R}$ in (83); to insure convergence for all \vec{k} , one must require $\alpha > \kappa$. The result is continued down to $\alpha = 0$ after the sum is done). For our choice of $\phi(R)$, the calculation of $\phi(k)$ is trivial, and yields

$$\phi(k) = \frac{4\pi e^2}{k^2} \cos k a \quad (85)$$

(this is also to be understood in the sense of the limit of the result for a damped potential). We substitute (85) in (84) and make a change of variables

$$k = \kappa X \quad (86)$$

to obtain

$$\phi_D(R) = \frac{2e^2}{\pi R} \int_0^\infty \frac{dX X \sin \lambda X \cos \delta X}{[X^2 + \cos \delta X]} \quad (87)$$

where

$$\lambda = \kappa R \quad (88)$$

and

$$\delta = \kappa a = \sqrt{\epsilon} \quad (89)$$

We wish to approximate (87) in such a way as to give results correct through ϵ^2 to the quantities U/θ , where \underline{U} is the ensemble average of the N particle potential. We will show that this can be done by replacing the cosine in the denominator by the first two terms in a power series:

$$\phi_D^{(0)} = \frac{2e^2}{\pi R} \int_0^\infty \frac{dX X \sin \lambda X \cos \delta X}{1 + X^2 \left(1 - \frac{\delta^2}{2}\right)} \quad (90)$$

Since the integrand is an even function of X , we may replace the integral by $1/2$ the integral from $(-\infty, \infty)$ and use the method of residues.

The calculation is straightforward and gives

$$\begin{aligned}
\phi_D^{(0)} &= \frac{e^2}{R \left(1 - \frac{\delta^2}{2}\right)} \left\{ S(\lambda - \delta) e^{-\frac{\lambda}{\sqrt{1 - \frac{\delta^2}{2}}}} \cosh \frac{\delta}{\sqrt{1 - \frac{\delta^2}{2}}} \right. \\
&\quad \left. - S(\delta - \lambda) e^{-\frac{\delta}{\sqrt{1 - \frac{\delta^2}{2}}}} \sinh \frac{\lambda}{\sqrt{1 - \frac{\delta^2}{2}}} \right\} \\
&\approx \frac{e^2}{R} \left\{ S(R - a) e^{-\kappa R} \left[1 + \epsilon - \frac{\kappa R \epsilon}{4} + O(\epsilon^2) \right] \right. \\
&\quad \left. - S(a - R) (\kappa R) \left[1 - \sqrt{\epsilon} + \frac{5\epsilon}{4} + \frac{(\kappa R)^2}{6} \right. \right. \\
&\quad \left. \left. + O\left(\epsilon^{\frac{3}{2}}\right) \right] \right\} \quad (91)
\end{aligned}$$

(here we have included all terms through ϵ^2 in $\phi_D^{(0)}/\theta$.)

The remainder integral is

$$\phi_D^{(1)} = \phi_D - \phi_D^{(0)} = \frac{e^2}{\pi R} \int_{-\infty}^{\infty} \frac{dz \, z \sin \lambda z \cos \delta z \left[1 - \frac{(\delta z)^2}{2} - \cos \delta z \right]}{\left[1 + z^2 \left(1 - \frac{\delta^2}{2} \right) \right] \left[z^2 + \cos \delta z \right]} \quad (92)$$

But

$$z^2 + \cos \delta z \geq z^2 \left(1 - \frac{\delta^2}{2} \right) + 1 \geq 0 \quad (93)$$

so

$$\left| \phi_D^{(1)} \right| \leq \frac{e^2}{\pi R} \left| \int_{-\infty}^{\infty} dz \frac{z \sin \lambda z \cos \delta z \left[1 - \frac{(\delta z)^2}{2} - \cos \delta z \right]}{\left[z^2 \left(1 - \frac{\delta^2}{2} \right) + 1 \right]^2} \right| \quad (94)$$

The integral can again be evaluated by residues; rather than quote the rather lengthy result, we merely state that it has the form

$$\left| \phi_D^{(1)} \right| \leq \theta \left| S(R - 2a) e^{-\kappa R} g(R) + S(2a - R) h(R) \right| \quad (95)$$

where

$$g(R) = O\left(\epsilon^{\frac{5}{2}}\right); \quad h(R) = O\left(\epsilon^{\frac{3}{2}}\right) \quad (96)$$

The long range part is clearly negligible for our purposes; for the short range part, we observe that in calculating U/θ , we introduce another factor of the potential and integrate over a small volume of radius $2a$; thus we will have an additional factor of

$$\frac{1}{v\theta} \int_{R < 2a} dR \left(\frac{e^2}{R} \right) = \frac{4\pi(2a)^2}{2v\theta} \approx \epsilon \quad (97)$$

and the contribution of \underline{h} will also be negligible.

It will be observed that the leading term in ϕ_D at long range is just the Debye potential as expected. At short range, the significant term is just the negative of the Coulomb potential of particles separated by a Debye length. Thus the "screening" effect produces at short range a weak attractive effective potential (however the repulsive ψ potential dominates).

7. The Average Energy: Equation of State and Thermodynamic Quantities

We now turn to the computation of the thermodynamic properties for the electron gas. The simplest of these to calculate is the average interaction energy:

$$U = \frac{\int \dots \int d\mathbf{R}_1 \dots d\mathbf{R}_N e^{-(\Psi_N + \Phi_N)/\theta}}{\int \dots \int d\mathbf{R}_1 \dots d\mathbf{R}_N e^{-(\Phi_N + \Psi_N)/\theta}} - U_+ \quad (98)$$

where U_+ represents the effect of the neutralizing background. In terms of the two particle distribution this may be written as

$$U = \frac{N}{2v} \int d\mathbf{R} \frac{e^2}{R} (n_2(R) - 1) = \frac{N}{2v} \int d\mathbf{R} (\psi(R) + \phi(R)) \left[e^{-\psi(R)/\theta} m_2(R) - 1 \right] \quad (99)$$

We wish to use the results of the preceding sections to calculate U/θ through second order in the plasma parameter. We will separately calculate the contribution of each term of Section 5, and label the corresponding contributions to \underline{U} with the same numbers. We start with the contribution of $m_s^{(0)} = 1$:

$$\begin{aligned} U_0 &= \frac{N}{2v} \int d\mathbf{R} (\psi(R) + \phi(R)) \left[e^{-\psi(R)/\theta} m_s^{(0)} - 1 \right] \\ &= \left(\frac{N\theta}{2} \right) \left(\frac{4\pi e^2}{\theta v} \right) \int_0^a dR R \left(e^{-\frac{e^2}{\theta R}} - 1 \right) \end{aligned} \quad (100)$$

The integral may be evaluated approximately by successive integration by parts:

$$U_0 = \frac{N\theta}{2} \kappa^2 \left\{ \frac{a^2}{2} \left(e^{-\frac{e^2}{\theta a}} - 1 \right) - \frac{e^2 a}{2\theta} e^{-\frac{e^2}{\theta a}} + \frac{e^2}{2\theta^2} \int_0^a \frac{dR e^{-\frac{e^2}{\theta R}}}{R} \right\} \quad (101)$$

We recall that

$$\kappa a = e^2 / \theta a = \sqrt{\epsilon} \quad , \quad (102)$$

so the exponentials may be expanded; in the integral term we substitute

$$R = e^2 / (\theta z) \quad . \quad (103)$$

Then

$$\begin{aligned} U_0 &= \frac{N\theta}{2} \left\{ -\epsilon^{\frac{3}{2}} + \frac{3\epsilon^2}{4} + O\left(\epsilon^{\frac{5}{2}}\right) \right. \\ &\quad \left. + \frac{\epsilon^2}{2} \int_{\sqrt{\epsilon}}^{\infty} \frac{dz e^{-z}}{z} \right\} \\ &= \frac{N\theta}{2} \left\{ -\epsilon^{\frac{3}{2}} + \epsilon^2 \left[\frac{3}{4} - \frac{1}{2}(\gamma + \frac{1}{2} \log \epsilon) \right] \right. \\ &\quad \left. + O\left(\epsilon^{\frac{5}{2}}\right) \right\} \end{aligned} \quad (104)$$

where we have used the familiar result

$$\int_{\delta}^{\infty} dz \frac{e^{-z}}{z} = -(\gamma + \log \delta) + o(\delta) \quad (105)$$

where γ is the Euler-Mascheroni constant.

We will now show that the contribution of the short range terms $m_2^{(1,0)}$ and $m_2^{(2,0)}$ is negligible (i.e. $o(\epsilon^2)$). From (54), (58), (11), (12) and (72),

$$\begin{aligned} U_{1,0} &= \frac{N}{2v^2} \int d\tilde{R} \frac{e^2}{R} \left[e^{-\frac{e^2}{\theta R}} S(a - R) + S(R - a) \right] \\ &\int d\tilde{R}' \left(e^{-\frac{e^2}{\theta R'}} - 1 \right) \left(e^{-\frac{e^2}{\theta |\tilde{R} - \tilde{R}'|}} - 1 \right) S(a - R') S(a - |\tilde{R} - \tilde{R}'|) \\ &= \frac{4\pi^2 N e^2}{v^2} \int_0^{\infty} dR \left[e^{-\frac{e^2}{\theta R}} S(a - R) + S(R - a) \right] \\ &\int_0^a dR' R' \left(e^{-\frac{e^2}{\theta R'}} - 1 \right) \int_{|R - R'|}^{(R + R')} dz z S(a - z) \left(e^{-\frac{e^2}{\theta z}} - 1 \right) \quad (106) \end{aligned}$$

For purposes of estimation we may replace the square bracket (which is always less than one) by unity and integrate by parts in the \underline{R} integral to obtain

$$|U_{1,0}| < \frac{N \theta \pi \kappa^2}{v} \left| \int_0^{\infty} dR R \int_0^a dR' R' \left(e^{-\frac{e^2}{\theta R'}} - 1 \right) \right|$$

$$\left[(R + R') S(a - R - R') \left(e^{-\frac{e^2}{\theta(R + R')}} - 1 \right) - (R - R') S(a - |R - R'|) \left(e^{-\frac{e^2}{\theta|R - R'|}} - 1 \right) \right] \quad (107)$$

We first consider the terms involving $R + R'$; since

$$R + R' \geq R' \geq 0$$

$$\left(1 - e^{-\frac{e^2}{\theta(R + R')}} \right) \geq \left(1 - e^{-\frac{e^2}{\theta R'}} \right) \geq 0 \quad (108)$$

so the contribution of the first term in (107) may be estimated as

$$|U_{1,0}^{(1)}| < \frac{N\theta\pi\kappa^2}{v} \int_0^a dR R \int_0^{a-R} dR' R' (R + R') \left(e^{-\frac{e^2}{\theta R'}} - 1 \right)^2 \quad (109)$$

The integral may be done by integration by parts, similar to the derivation of (104); the result is

$$|U_{1,0}^{(1)}| < \frac{N\theta}{4} \epsilon^{\frac{5}{2}} \left[\frac{3}{4} + \frac{2}{3} + \frac{2}{3} \log 2 + \frac{1}{6} \log \epsilon + O(\sqrt{\epsilon \log \epsilon}) \right] \quad (110)$$

The terms involving $|R - R'|$ are somewhat more difficult, but one can show that they are the same order. Thus

$$U_{1,0} = o(\epsilon^2) \quad (111)$$

While this result may seem rather surprising, it can be understood qualitatively as follows: While one might expect $m_2^{(1,0)}$ to be $O(\sqrt{\epsilon})$, for the particular ψ chosen here each factor of $f(R)$ effectively contributes another factor of $\sqrt{\epsilon}$ (or possibly $\sqrt{\epsilon} \log \epsilon$) on integration. The factor of the potential gives another $\sqrt{\epsilon}$, and the short range of $m_2^{(1,0)}$ produces a factor of $\frac{4\pi a^3}{v} = \sqrt{\epsilon}$. This suggests a total contribution of order $\epsilon^{\frac{5}{2}}$ or $\epsilon^{\frac{5}{2}} \log \epsilon$, in agreement with (110). Since $m_2^{(2,0)}$ is less than $m_2^{(1,0)}$ by a factor of $\sqrt{\epsilon}$ and is equally short range, one also has

$$U_{2,0} = o(\epsilon^2) \quad (112)$$

We now turn to the terms involving ϕ_D . From (45),

$$U_{0,1} = -\frac{4\pi N e^2}{2v\theta} \int_0^\infty dR R \phi_D(R) e^{-\psi(R)/\theta} \quad (113)$$

For ϕ_D we substitute the expression (91); the integrals are straightforward and give

$$U_{0,1} = \frac{N\theta}{2} \left[-\epsilon + \epsilon^{\frac{3}{2}} - \frac{3\epsilon^2}{4} + o\left(\epsilon^{\frac{5}{2}}\right) \right] \quad (114)$$

The second order contribution is

$$U_{0,2} = \frac{N}{4v} \int d\tilde{R} \left[\psi(R) e^{-\psi(R)/\theta} + \phi(R) \right] \left\{ \left(\frac{\phi_D(R)}{\theta} \right)^2 \right.$$

$$\begin{aligned}
& - \frac{2}{v\theta^3} \int d\underline{R}' \phi_D^2(\underline{R}') \phi_D(|\underline{R} - \underline{R}'|) \\
& + \frac{1}{v^2 \theta^4} \int d\underline{R}' \int d\underline{R}'' \left(\phi_D(\underline{R}') \right)^2 \phi_D(\underline{R}'') \phi_D(|\underline{R} - \underline{R}' + \underline{R}''|) \Big\} .
\end{aligned}
\tag{115}$$

It is easy to show that the integrals involving ψ are negligible, largely because of their short range. For the others, we use the facts that

$$\begin{aligned}
& \int d\underline{R}' \phi(\underline{R}') \phi_D(|\underline{R} - \underline{R}'|) = \\
& - v\theta \left[\phi_D(\underline{R}) - \phi(\underline{R}) \right]
\end{aligned}
\tag{116}$$

and

$$\begin{aligned}
& \int d\underline{R}' \phi_D(\underline{R}') \phi_D(|\underline{R} - \underline{R}'|) = \\
& - \frac{v\theta\kappa}{2} \frac{\partial}{\partial \kappa} \phi_D(\underline{R}) \Big|_{e^2, R, a, \text{const}},
\end{aligned}
\tag{117}$$

which follow from the definition (81) of ϕ_D as a sum of convolutions of ϕ . The use of (116), (117) in (115) leads to

$$\begin{aligned}
U_{0,2} & \approx \frac{N}{4v\theta^2} \int d\underline{R} \left(\phi_D(\underline{R}) \right)^2 \left[\phi_D(\underline{R}) + \frac{\kappa}{2} \frac{\partial}{\partial \kappa} \phi_D(\underline{R}) \right] \\
& = \frac{N\theta}{4} \left[-\epsilon^2 \left(\gamma + \log 3\sqrt{\epsilon} \right) \right. \\
& \quad \left. - \frac{\epsilon^2}{6} + O\left(\epsilon^{\frac{5}{2}}\right) \right]
\end{aligned}
\tag{118}$$

Where we have used (91) and (105).

Finally, we calculate the contribution of the "cross term", $m_2^{(1,1)}$. Once again we can neglect the short range potential ψ in (99); for similar reasons, any terms in (71) containing two factors of "f" gives a negligible contribution. On the remaining terms we again use (116), (117), to obtain

$$\begin{aligned} U_{1,1} &\approx \frac{N}{2v} \int dR \ f(R) \left[\phi_D(R) + \frac{\kappa}{2} \frac{\partial}{\partial \kappa} \phi_D(R) \right] \\ &= - \frac{3N\theta}{4} \left[\kappa^3 \int_0^a dR R^2 \left(e^{-\frac{e^2}{\theta R}} - 1 \right) + o\left(\epsilon^{\frac{5}{2}}\right) \right] \\ &= \frac{3N\theta}{8} \left[\epsilon^2 + o(\epsilon^2) \right] \end{aligned} \quad (119)$$

Adding the results (104), (111), (112), (114), (118) and (119), we have

$$U = \frac{N\theta}{2} \left[-\epsilon + \epsilon^2 \left(-\gamma - \frac{1}{2} \log 3\epsilon + \frac{2}{3} \right) + o(\epsilon^2) \right] \quad (120)$$

Since most authors calculate the Helmholtz free energy instead of \underline{U} , we will calculate it from the formula

$$F = F_0 + \int_0^1 \frac{d\lambda U(\lambda)}{\lambda}, \quad (121)$$

where F_0 is the perfect gas free energy and $U(\lambda)$ is the function obtained by replacing e^2 everywhere by $e^2\lambda$. From the definition of ϵ, κ , we may obtain $U(\lambda)$ from (120) by the substitution

$$\epsilon \rightarrow \lambda^{\frac{3}{2}} \epsilon$$

The integral in (121) is then straightforward, and we find

$$(F - F_0) = \frac{N\theta}{3} \left\{ -\epsilon + \frac{\epsilon^2}{2} \left[-\gamma - \frac{1}{2} \log 3\epsilon + \frac{11}{12} \right] + o(\epsilon^2) \right\} \quad (122)$$

From (122), one may derive all the thermodynamic quantities; in particular, the equation of state is given by

$$p = - \frac{\partial F}{\partial V} = \frac{\theta}{v} \left\{ 1 - \frac{\epsilon}{6} + \frac{\epsilon^2}{6} \left[-\gamma - \frac{1}{2} \log 3\epsilon + \frac{2}{3} \right] + o(\epsilon^2) \right\}, \quad (123)$$

which could also have been derived from the relation

$$p = \frac{\theta}{v} + \frac{U}{3v} \quad (124)$$

The results (122), (123), are identical with those obtained by Abe² and Friedman³, using diagrammatic techniques.

8. Summary and Discussion

We have developed a general method for treating systems whose interaction consists of two parts, one of which is short range, and the other weak. It has further been shown that such a scheme may be used to describe an electron gas in a neutralizing background. provided the plasma

parameter is small. Finally, explicit results have been obtained for the thermodynamic quantities, valid through second order in the plasma parameter. This is believed to be the first such derivation based on the nature of the potential rather than on topological considerations. In a subsequent paper, we show how the method may be generalized to handle nonequilibrium situations. In particular, we will derive a kinetic equation which is not subject to the usual short range divergence.

FOOTNOTES

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